Solutions of final exam 1402

■ **مسئله اول: (۱۰+۱۰ نمره.)** دو حالت دو کیوبیتی زیر را در نظر بگیرید:

$$
\begin{aligned}
\hat{\rho} &= \frac{1}{4}(I + x\sigma_x \otimes \sigma_x - x\sigma_y \otimes \sigma_y + z\sigma_z \otimes \sigma_z) \\
\hat{\sigma} &= \frac{1}{4}(I + x\sigma_x \otimes \sigma_x - x\sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z) \qquad (۱)
\end{aligned}
$$

فاصله این دو حالت و هم چنین شباهت آنها را با هم حساب کنید.

We first calculate the distance. $\hat{\rho} - \hat{\sigma} = \frac{1}{4} \sum_{i=1}^{3} (r_i - s_i)\, \sigma_i \otimes \sigma_i$

$$\rightarrow \hat{\rho} - \hat{\sigma} = \frac{1}{4} \begin{vmatrix} z-1 & \cdot & \cdot & \cdot \\ \cdot & 1-z & \cdot & \cdot \\ \cdot & \cdot & 1-z & \cdot \\ \cdot & \cdot & \cdot & z-1 \end{vmatrix}$$

where $\eta_i = r_i - s_i$.

$$\lambda(\hat{\rho} - \hat{\sigma}) = \frac{1}{4}\{z-1,\ 1-z,\ 1-z,\ z-1\} \rightarrow$$

$$D(\hat{\rho}, \hat{\sigma}) = \frac{1}{2}|z-1|$$

$F(\rho, \sigma) = ?$ Both ρ and σ are in block diagonal form: $\rho = \rho_1 \oplus \rho_2$

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$$\rho = \frac{1}{4} \begin{vmatrix} 1+z & & & 2x \\ & \boxed{\begin{matrix} 1-z & 0 \\ 0 & 1-z \end{matrix}} & \\ 2x & & & 1+z \end{vmatrix} \qquad \sigma = \frac{1}{4} \begin{vmatrix} 2 & & & 2x \\ & \boxed{\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}} & \\ 2x & & & 2 \end{vmatrix}$$

ρ_1 $\qquad\qquad$ σ_1

therefore from the basic definition, $\rightarrow$

$$F(\rho,\sigma) = \mathrm{tr}\sqrt{\rho^{1/2}\sigma\rho^{1/2}} = \mathrm{tr}\sqrt{(\rho_1\oplus\rho_2)^{1/2}(\sigma_1\oplus\sigma_2)(\rho_1\oplus\rho_2)^{1/2}}$$

$$= \mathrm{tr}\sqrt{(\rho_1^{1/2}\oplus\rho_2^{1/2})(\sigma_1\oplus\sigma_2)(\rho_1^{1/2}\oplus\rho_2^{1/2})} = \mathrm{tr}\sqrt{(\rho_1^{1/2}\sigma_1\rho_1^{1/2})\oplus(\rho_2^{1/2}\sigma_2\rho_2^{1/2})}$$

$$= \mathrm{tr}\sqrt{\rho_1^{1/2}\sigma_1\rho_1^{1/2}} + \mathrm{tr}\sqrt{\rho_2^{1/2}\sigma_2\rho_2^{1/2}} = F(\rho_1,\sigma_1) + F(\rho_2,\sigma_2)$$

Since $\sigma_1 = 0 \rightarrow$ $F(\rho,\sigma) = F(\rho_2,\sigma_2)$

the outer blocks.

From the identity $(\sqrt{\lambda_1}+\sqrt{\lambda_2})^2 = \lambda_1+\lambda_2+2\sqrt{\lambda_1\lambda_2} \rightarrow$

$$(\mathrm{tr}\sqrt{A})^2 = \mathrm{tr}\,A + 2\sqrt{\det A} \longrightarrow \mathrm{tr}\sqrt{A} = \sqrt{\mathrm{tr}(A) + 2\sqrt{\det A}}$$

$$\rightarrow F(\rho_2,\sigma_2) = \sqrt{\mathrm{tr}(\rho_2\sigma_2) + 2\sqrt{\det(\rho_2)\det(\sigma_2)}}$$

$$\rightarrow \boxed{F(\rho,\sigma) = \frac{1}{4}\sqrt{4(1+z) + 8x^2 + 2\sqrt{[(1+z)^2 - 4x^2][4-4x^2]}}}$$

■ **مسئله دوم: (۲۰ نمره.)** ظرفیت کلاسیک کانال کوانتومی واقطبش را برای کیوبیت ها حساب کنید. این کانال به شکل زیر تعریف می شود:

$$\mathcal{E}(\rho) = (1-p)\rho + \frac{p}{2}I. \qquad (۲)$$

$\mathcal{E}(\rho) = (1-p)\rho + p\,\frac{I}{d}$

this channel is covariant: $\mathcal{E}(g\rho g^\dagger) = g\,\mathcal{E}(\rho)\,g^\dagger \quad \forall g \in SU(d)$

We have the following formula:

$$C_{cl}(\mathcal{E}) = \max_{\{\rho_i, p_i\}} \chi(\{\rho_i, p_i\}) = \max_{\{\rho_i, p_i\}} \left[S\left(\mathcal{E}\left(\sum_i p_i \rho_i\right)\right) - \sum_i p_i\, S(\mathcal{E}(\rho_i)) \right]$$

Let $p_i = \frac{1}{|G|}$ and $\rho_i = g_i \rho g_i^\dagger$ where $g_i \in G$ the covariance group. $\rightarrow$ $\rho = \sum_i p_i \rho_i = \frac{1}{|G|}\sum_i g_i \rho g_i^\dagger \rightarrow \rho g_j = g_j \rho \quad \forall g_i$

From Schur's Lemma: $\rightarrow \rho \propto I \rightarrow$ Since $tr(\rho) = 1 \rightarrow \rho = \frac{1}{d} I \rightarrow$

$$S\left(\mathcal{E}\left(\sum_i p_i \rho_i\right)\right) = S\left(\mathcal{E}\left(\frac{1}{|G|}\sum_i g_i \rho g_i^\dagger\right)\right) = S\left(\mathcal{E}\left(\frac{I}{d}\right)\right) = S\left(\frac{I}{d}\right) = \log d.$$

$\rightarrow$ the first term is maximized. $\rightarrow$ $C_{cl}(\mathcal{E}) = \log d - \min \sum_i \frac{1}{|G|} S(\mathcal{E}(g_i \rho g_i^\dagger))$

$$= \log d - \text{Min} \sum_i \frac{1}{|G|} S(g_i \cdot \mathcal{E}(\rho_0) g_i^\dagger) \xrightarrow{\text{Covariance}}$$

$$= \log d - \text{Min}\ S(\mathcal{E}(\rho_0)).$$

From convexity of $S \rightarrow$ ρ_0 can be chosen to be pure. $\rightarrow$

$SU(d)$ invariance $\rightarrow$ we can take $\rho_0 = |0\rangle\langle 0|$ $\rightarrow$

$$\mathcal{E}(|0\rangle\langle 0|) = (1-p)|0\rangle\langle 0| + \frac{p}{d} I = \begin{vmatrix} 1-p+\frac{p}{d} & & & \\ & \frac{p}{d} & & \\ & & \ddots & \\ & & & \frac{p}{d} \end{vmatrix}$$

$$S(\mathcal{E}(|0\rangle\langle 0|)) = -\left\{ \left(1-p+\frac{p}{d}\right) \log_2 \left(1-p+\frac{p}{d}\right) + \frac{d-1}{d} p \log \frac{p}{d} \right\}$$

$$\rightarrow \boxed{C_{cl}(\mathcal{E}_{dep}) = \log_2 d + \left(1-p+\frac{p}{d}\right) \log_2 \left(1-p+\frac{p}{d}\right) + \frac{d-1}{d} p \log \frac{p}{d}}$$

For $p \rightarrow 0$ $\quad C_{cl}(\mathcal{E}_{dp}) = \log_2 d$ ✓

For $p \rightarrow 1$ $\quad C_{cl}(\mathcal{E}_{dep}) = \log_2 d + \frac{1}{d} \log_2 \frac{1}{d} + \frac{d-1}{d} \log \frac{1}{d} = 0$ ✓

■ **مسئله سوم: (۲۰ نمره.)** کانال میراکننده دامنه را برای کیوبیت ها در نظر گرفته و کیفیت آن را حساب کنید. این کانال به شکل زیر تعریف می شود::

$$\Lambda(\rho) = A_0\rho A_0^\dagger + A_1\rho A_1^\dagger \tag{۳}$$

که در آن

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & \cos\theta \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & \sin\theta \\ 0 & 0 \end{pmatrix}, \tag{۴}$$

کیفیت یک کانال نیز چنین تعریف می شود:

$$Q(\Lambda) := Min_\psi F(|\psi\rangle, \Lambda(|\psi\rangle\langle\psi|)) \tag{۵}$$

$$\mathcal{E}_{AD}(\rho) = A_0 \rho A_0^\dagger + A_1 \rho A_1^\dagger \quad \text{where} \quad A_0 = \begin{bmatrix} 1 & 0 \\ 0 & \cos\theta \end{bmatrix} \quad A_1 = \begin{bmatrix} 0 & \sin\theta \\ 0 & 0 \end{bmatrix}$$

$$Q(\mathcal{E}_{AD}) = \min_{\psi} F(|\psi\rangle, \mathcal{E}_{AD}(|\psi\rangle\langle\psi|))$$

$$= \min_{\psi} \left(\langle\psi| A_0|\psi\rangle\langle\psi| A_0^\dagger|\psi\rangle + \langle\psi| A_1|\psi\rangle\langle\psi| A_1^\dagger|\psi\rangle \right)$$

Let $|\psi\rangle = \begin{pmatrix} \cos\alpha \\ \sin\alpha\, e^{i\beta} \end{pmatrix} \rightarrow$ For simplicity let $\cos\alpha =: c$ $\sin\alpha =: s$

$$Q = \min_{\alpha,\beta} \left[\left| (c \;\; s e^{-i\beta}) \begin{pmatrix} 1 & 0 \\ 0 & \cos\theta \end{pmatrix} \begin{pmatrix} c \\ s e^{i\beta} \end{pmatrix} \right|^2 + \left| (c \;\; s e^{-i\beta}) \begin{bmatrix} 0 & \sin\theta \\ 0 & 0 \end{bmatrix} \begin{pmatrix} c \\ s e^{i\beta} \end{pmatrix} \right|^2 \right.$$

$$= \min_{\alpha,\beta} \left\{ |c^2 + s^2 \cos\theta|^2 + |s^2 \sin\theta|^2 \right\}$$

$$= \min_{\alpha,\beta} \left\{ |\cos^2\alpha + \sin^2\alpha \cos\theta|^2 + |\sin^2\alpha \sin\theta|^2 \right\}$$

$$= \min_{\alpha} \left\{ \cos^4\alpha + \sin^4\alpha + 2\sin^2\alpha \cos^2\alpha \cos^2\theta \right\}$$

$$= \min_{\alpha} \left\{ 1 + 2\sin^2\alpha \cos^2\alpha (\cos^2\theta - 1) \right\} = \min_{\alpha} \left\{ 1 + \frac{1}{2}\sin^2 2\alpha (\cos^2\theta - 1) \right\}$$

Since $\cos^2\theta - 1 < 0 \rightarrow$ the minimum is obtained when we set $\sin^2 2\alpha = 1$,

$$= 1 + \frac{1}{2}(\cos^2\theta - 1) = \frac{1}{2}(1 + \cos^2\theta). \longrightarrow Q(\mathcal{E}_{AD}) = \frac{1}{2}(1 + \cos^2\theta)$$

■ **مسئله چهارم: (۱۰+ ۱۰ نمره)** الف- نشان دهید که اگر اگر کانال Λ رد نگه دار باشد، کانال مکمل آن یعنی Λ^c نیز ردّ نگه دار است.

ب: در بعد d یک کانال کوانتومی به شکل زیر تعریف شده است:

$$\mathcal{E}(\rho) = Tr(\rho)\frac{I}{d}. \qquad (۶)$$

مکمل این کانال را بسازید.

We know that: if $\mathcal{E}(\rho) = \sum_{\alpha} A_\alpha \rho A_\alpha^\dagger \rightarrow$

$\mathcal{E}^c(\rho) = \sum_i K_i \rho K_i^\dagger$ in which $(K_i)_{\alpha j} = (A_\alpha)_{ij}$

الف - we will show that $\sum_i K_i^\dagger K_i = I$

we calculate the elements: $\left(\sum_i K_i^\dagger K_i\right)_{jk} =$

$$\sum_{i,\alpha} (K_i^\dagger)_{j\alpha} (K_i)_{\alpha k} = \sum_{i,\alpha} \overline{(K_i)_{\alpha j}} (K_i)_{\alpha k} =$$

$$= \sum_{i,\alpha} \overline{(A_\alpha)_{ij}} (A_\alpha)_{ik} = \sum_{i,\alpha} (A_\alpha^\dagger)_{ji} (A_\alpha)_{ik}$$

$$= \sum_\alpha (A_\alpha^\dagger A_\alpha)_{jk} = (I)_{jk} = \delta_{jk} \checkmark.$$

Now $\mathcal{E}(\rho) = \mathrm{Tr}(\rho)\frac{I}{d} = \frac{1}{d} \sum_{ij} \langle i|\rho|i\rangle |j\rangle\langle j|$

$\to \mathcal{E}(\rho) = \frac{1}{d} \sum_{ij} |j\rangle\langle i|\rho|i\rangle\langle j|$

$\to \mathcal{E}(\rho) = \sum_{ij} A_{ij} \rho A_{ij}^\dagger$ where $A_{ij} = \frac{1}{\sqrt{d}} |j\rangle\langle i|$

So the Kraus operators of $\mathcal{E}$ have double indices. $\to$

$\mathcal{E}^c(\rho) = \sum_k K_k \rho K_k^\dagger$ where $(K_k)_{ij,m} = (A_{ij})_{k,m} \Rightarrow$

$$(K_k)_{ij,m} = \langle k| A_{ij} |m\rangle = \frac{1}{\sqrt{d}} \langle k|j\rangle\langle i|m\rangle = \frac{1}{\sqrt{d}} \delta_{kj} \delta_{im}.$$

$$\rightarrow \quad \mathcal{E}^c(\rho) = \sum_k K_k \rho K_k^\dagger \longrightarrow$$

$$[\mathcal{E}^c(\rho)]_{ij,pq} = \sum_{\substack{k \\ m,n}} (K_k)_{ij,m} \, \rho_{mn} \, (K_k^\dagger)_{n,pq}$$

$$= \sum_{\substack{k \\ m,n}} (K_k)_{ij,m} \, \rho_{mn} \, (K_k)^*_{pq,n}$$

$$= \frac{1}{d} \sum_{k,m,n} \delta_{kj} \cdot \delta_{im} \, \rho_{mn} \, \delta_{kq} \, \delta_{pn}$$

$$= \frac{1}{d} \rho_{ip} \, \delta_{jq} = \frac{1}{d} (\rho \otimes I)_{ij,pq}$$

$$\rightarrow \boxed{\mathcal{E}^c(\rho) = \frac{1}{d} \rho \otimes I}$$